

Universal R operator with deformed conformal symmetry

D. Karakhanyan^{a,b}, R. Kirschner^b and M. Mirumyan^a

^a Yerevan Physics Institute,
Br. Alikhanian st. 2, 375036 Yerevan, Armenia.

^b Naturwissenschaftlich-Theoretisches Zentrum und Institut für Theoretische Physik,
Universität Leipzig,
Augustusplatz 10, D-04109 Leipzig, Germany

Abstract

We study the general solution of the Yang-Baxter equation with deformed $sl(2)$ symmetry. The universal R operator acting on tensor products of arbitrary representations is obtained in spectral decomposition and in integral forms. The results for eigenvalues, eigenfunctions and integral kernel appear as deformations of the ones in the rational case. They provide a basis for the construction of integrable quantum systems generalizing the XXZ spin models to the case of arbitrary not necessarily finite-dimensional representations on the sites.

1 Introduction

All known exactly solved many-body models belong to two large classes: two-dimensional classical statistical models (Ising model, vertex models [1]) and one-dimensional quantum models. The standard example of one-dimensional quantum model is Heisenberg spin chain which has been studied in much detail for finite-dimensional representations. The method of solving the Schrödinger equation for this model was pioneered by Bethe [2]. Later it has been understood that the Bethe ansatz can be applied to models, in which the scattering between (quasi-) particles is purely elastic. Mathematically the condition of elasticity is expressed by Yang-Baxter equation.

The generalization of the Bethe ansatz solution, the Quantum Inverse Scattering Method, was proposed by the group headed by L. D. Faddeev [3].

The simplest $SU(2)$ -symmetric spin $1/2$ Heisenberg chain was generalized for the cases of lower spin to the uniaxial $U(1)$ -symmetric systems [4] and to biaxial spin systems [1] and to the case of higher spins [5].

A construction of the universal R-matrix acting on arbitrary spin $(\frac{n}{2}, \frac{m}{2})$ representations of finite dimensions relies on the fusion procedure [6]. Based on the result in [7] the models of homogeneous periodic XXZ chains of arbitrary spins $(\frac{n}{2} = \frac{m}{2})$ have been studied, working out Hamiltonians, Bethe ansatz and thermodynamic properties.

Establishing some first concepts of quantum groups Jimbo [8] derived the universal XXZ R operator in algebraic terms in the spectral decomposition form. The deformed $sl(2)$ algebra relations appeared first in [9].

The methods of quantum groups [10, 8, 11, 12] allow a calculation in the framework of the q -deformed universal enveloping algebra of the loop group $sl(\hat{2})$. This has been performed in [13] where the result is given in an algebraic form in terms of a series. The result has been used in [14] in the study of the integrable structure of conformal field theory. In [15] a method of constructing trigonometric R operators in spectral decomposition form has been developed which is applicable to tensor products of any affinizable representations of quantum algebras and superalgebras.

The general form of L operators (finite dimensional representations) intertwined by the XXZ fundamental representation R matrix for the case of the deformation parameter being a root of unity has been given in [16] explaining the relation to the chiral Potts model. The universal R operator has been constructed for this case.

In recent years high energy scattering in gauge field theories has been discovered [17] as a new field of applications of integrable quantum systems. In a number of cases the leading contribution to the effective interaction appearing in the Regge or the Bjorken asymptotics of scattering is determined by Hamiltonians of integrable chains. Chains with a few sites are of interest describing the contribution of the exchange of the corresponding number of partons or reggeons. Unlike the mostly studied spin chains however the quantum spaces corresponding to the sites are infinite dimensional involving all the momentum states of the partons (longitudinal momentum, one dimensional) or reggeons (transverse momentum, two dimensional). Also the application of the Bethe ansatz technique is not straightforward [18].

Therefore it is necessary to adapt the formulation and representation of the known methods of quantum integrable systems to the special needs of the new applications. A number of activities are going in this direction [19, 20, 21, 22, 23, 24, 25, 26]. The problems and the viewpoints arising from high energy scattering motivate investigations being of general interest.

In this context it is convenient to represent the quantum state of a site as a wave function. In the case of partons (Bjorken limit) it would depend on a one dimensional position variable, in the case of reggeons on a two dimensional position. The symmetry algebra acts in terms of differential operators on the wave functions classified in the corresponding representations. Operators are conveniently represented in differential or integral forms.

Owing to applications to the Bjorken asymptotics of QCD [27, 28] in this way the general rational solution of the Yang-Baxter equation with $sl(2)$ and with $sl(2|1)$ symmetry have been considered [29, 30]. The algebra $sl(2)$ represents the one-dimensional conformal symmetry of the asymptotic QCD interaction. In particular it generates the Möbius transformations on the argument of the wave function. The algebra $sl(2|1)$ represents the superconformal symmetry playing the analogous role in the Bjorken asymptotics in $N = 1$ supersymmetric Yang-Mills theory.

A scheme emerges which because of the physical background is simple and clear. It is

based on the well known methods [31, 9, 32, 33, 34] and results in formulations suitable for the above mentioned applications. In the present paper we treat along this line the universal R operator in the situation of deformed $sl(2)$ symmetry. Presently we are not able to indicate the role which the deformed conformal symmetry in high energy scattering might play.

We obtain the action of the R operator on the tensor product of arbitrary representations of the deformed algebra of lowest weight type in spectral decomposition form by constructing the wave functions of lowest weight states in the irreducible decomposition with respect to the co-product generators and the corresponding eigenvalues. Furthermore, we obtain this universal R operator in integral form. Our results provide a basis for the construction of integrable systems generalizing the XXZ spin chain models to the case where the quantum spaces on the sites are arbitrary not necessarily finite-dimensional representations of the deformed $sl(2)$ algebra.

We restrict ourselves to generic values of the representation parameter. The representations considered are the ones constructed from a lowest weight vector. We shall not discuss here the peculiarities arising in the case of the deformation parameter being a root of unity, where cyclic representations are essential.

We summarize in Sect. 2 some results on the $sl(2)$ symmetric rational R operators in the form where representations are written in terms of wave functions, algebra generators represented as differential operators and the universal R operator is given in integral form. In Sect. 3 we recall necessary facts on the deformed algebra and consider in detail the functions appearing in the decomposition of the tensor product representation. In Sect. 4 we formulate the conditions on the universal R operator following from YBE, point out their equivalence to the basic intertwining property and obtain the eigenvalues determining the spectral decomposition of the universal R operator. The integral kernel of the R operator is derived in Sect. 5. In Sect. 6 we study the integrals appearing in the action of the obtained integral operator on the eigenfunctions.

2 The undeformed case

We summarize some basic relations of the $sl(2)$ symmetric (rational) solutions of the YBE in a formulation, where the representation spaces are the ones of polynomials and with the symmetry generators acting there as differential operators [29, 30]. This sets the convenient perspective on our approach.

The best known example of a solution of the YBE is given by the 4×4 matrix

$$R^{(-\frac{1}{2}, -\frac{1}{2})}(u) = \begin{pmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{pmatrix} \quad (2.1)$$

where $a = u + \eta$, $b = u$, $c = \eta$. u is the spectral parameter and η is the model parameter which can be set equal to unity. The relation to $sl(2)$ symmetry becomes evident in the expressions

$$R^{(-\frac{1}{2}, -\frac{1}{2})}(u) = u\hat{I} + \eta\hat{P} = \hat{L}^{(-\frac{1}{2})}(u + \frac{1}{2}; \eta) \quad (2.2)$$

in terms of the permutation operator of two spin- $\frac{1}{2}$ representation spaces, $\hat{P} = \frac{1}{2}(I \otimes I + \sigma^a \otimes \sigma^a)$, or in terms of the $\ell = -\frac{1}{2}$ representation Lax matrix, which in general looks like

$$\hat{L}^{(\ell)}(u) = uI_{2 \times 2} \otimes \hat{I} + \eta \sigma^a \otimes \hat{S}^{(\ell)a} = \begin{pmatrix} u + \eta S^0 & \eta S^- \\ \eta S^+ & u - \eta S^0 \end{pmatrix} \quad (2.3)$$

Here $S^{(\ell)a}$ are the $sl(2)$ generators

$$[S^0, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^0.$$

acting in the representation space $V^{(\ell)}$ and \hat{I} the identity operator in this space.

Moreover the particular YBE for two fundamental representations, $\ell_1 = \ell_2 = -\frac{1}{2}$, and one arbitrary, $\ell_3 = \ell$, with the substitution

$$R^{(-\frac{1}{2}, \ell)}(u) = \hat{L}^{(\ell)}(u + \frac{1}{2})$$

is equivalent to the commutation relations of $\hat{S}^{(\ell),a}$. Another particular representation case of YBE, with general ℓ_1, ℓ_2 and with $\ell_3 = -\frac{1}{2}$ can be written as

$$R_{12}^{(\ell_1, \ell_2)}(u-v) L_1^{(\ell_1)}(u) L_2^{(\ell_2)}(v) = L_2^{(\ell_2)}(v) L_1^{(\ell_1)}(u) R_{12}^{(\ell_1, \ell_2)}(u-v). \quad (2.4)$$

and involves the universal $sl(2)$ symmetric R-operator, i.e. the one acting on the tensor product of arbitrary representations $V^{(\ell_1)} \otimes V^{(\ell_2)}$. This relation allows to determine the latter from the given Lax operators [11].

The generic representation space $V^{(\ell)}$ can be chosen to be the space of polynomials in one variable x . Then the $sl(2)$ generators act as

$$\begin{aligned} S^{(\ell)-} &= S^{(0)-}, \quad S^{(\ell)0} = x^{-\ell} S^{(0)0} x^\ell, \quad S^{(\ell)+} = x^{-2\ell} S^{(0)+} x^{2\ell}. \\ S^{(0)-} &= \partial, \quad S^{(0)0} = x\partial, \quad S^{(0)+} = -x^2\partial, \end{aligned} \quad (2.5)$$

Notice that the information on the representation ℓ is mainly carried by the differential operators, whereas the representation spaces look alike. They are spanned by the monomials x^m and 1 represents the lowest weight vector annihilated by S^- . However, a symmetric scalar product and the corresponding normalization, $\langle x | \ell, m \rangle = C_{\ell, m} x^m$, depend unavoidably on the representation ℓ .

In the particular cases of negative half-integer ℓ , $2\ell = -N$, the rising operator \hat{S}^+ has a null vector, $\hat{S}^+ x^N = 0$, and $\{x^m\}_0^N$ spans the finite dimensional irreducible representation, corresponding to the representation of three-dimensional rotations of angular momentum $-\ell$.

The tensor product space $V^{(\ell_1, \ell_2)}$ is represented by polynomials in x_1 and x_2 . The action of the generators is given by (trivial co-product)

$$S^a = S_1^{(\ell_1)a} + S_2^{(\ell_2)a}. \quad (2.6)$$

The Clebsch-Gordan decomposition is readily recovered by noticing that

$$S^- (x_1 - x_2)^n = 0, \quad S^0 (x_1 - x_2)^n = (\ell_1 + \ell_2 + n) (x_1 - x_2)^n$$

and that the irreducible representations are spanned by

$$\phi_n^{(m)}(x_1, x_2) = (S^+)^m (x_1 - x_2)^n.$$

Now we consider $R_{12}^{(\ell_1, \ell_2)}$ as an operator acting on functions of x_1 and x_2 . Separating in (2.4) the dependence on $(u + v)$ and $(u - v)$ we obtain two sets of conditions,

$$R_{12}(u) S^a = S^a R_{12}(u), \quad (2.7)$$

and

$$R_{12}(u) \hat{K}(u) = \hat{K}(u) R_{12}(u), \quad (2.8)$$

where

$$\hat{K}(u) = \begin{pmatrix} \frac{u}{2}(S_2^0 - S_1^0) + S_1^0 S_2^0 + S_1^- S_2^+ & \frac{u}{2}(S_2^- - S_1^-) + S_1^0 S_2^- - S_1^- S_2^0 \\ \frac{u}{2}(S_2^+ - S_1^+) - S_1^0 S_2^+ + S_1^+ S_2^0 & \frac{u}{2}(S_1^0 - S_2^0) + S_1^0 S_2^0 + S_1^+ S_2^- \end{pmatrix},$$

and

$$\hat{\bar{K}}(u) = \begin{pmatrix} \frac{u}{2}(S_2^- - S_1^-) + S_1^0 S_2^0 + S_1^+ S_2^- & \frac{u}{2}(S_2^- - S_1^-) + S_1^- S_2^0 - S_1^0 S_2^- \\ \frac{u}{2}(S_2^+ - S_1^+) + S_1^0 S_2^+ - S_1^+ S_2^0 & \frac{u}{2}(S_1^0 - S_2^0) + S_1^0 S_2^0 + S_1^- S_2^+ \end{pmatrix}.$$

(2.7) expresses the $sl(2)$ symmetry. It implies that $\phi_n^{(m)}(x_1, x_2)$ are eigenfunctions of $\hat{R}_{12}(u)$ with the eigenvalues $R_n(u)$ independent on m . The matrices of operators $\hat{K}, \hat{\bar{K}}$ transform covariantly,

$$[\sigma^a - S^a, K(u)] = 0, \quad [\sigma^a + S^a, \bar{K}(u)] = 0. \quad (2.9)$$

Therefore the conditions (2.8) are not all independent and one of them is sufficient to determine the eigenvalues R_n . We choose the upper right corner component condition from (2.8) appearing as the simplest in our representation, calculate the action of the operators on $(x_1 - x_2)^n$ and arrive at the recurrence relation

$$R_n(u) = -R_{n-1}(u) \frac{u + n - 1 + \ell_1 + \ell_2}{-u + n - 1 + \ell_1 + \ell_2}, \quad (2.10)$$

with the known result

$$R_n(u) = (-1)^n R_0(u) \prod_{k=1}^n \frac{u + k - 1 + \ell_1 + \ell_2}{-u + k - 1 + \ell_1 + \ell_2}.$$

We would like to represent the R operator acting on polynomial functions $\psi(x_1, x_2)$ in integral form

$$R_{12}(u) \psi(x_1, x_2) = \int dx_{1'} dx_{2'} \mathcal{R}(x_1, x_2 | x_{1'}, x_{2'}) \psi(x_{1'}, x_{2'}) \quad (2.11)$$

The integration in $x_{1'}, x_{2'}$ is along closed contours. The defining relations result in a set of differential equations for the kernel. A set of four first order differential equations can be extracted and shown to imply the other equations. The result for the kernel is given by

$$\mathcal{R}(x_1, x_2 | x_{1'}, x_{2'}) = \frac{(x_1 - x_2)^\alpha (x_{1'} - x_{2'})^\delta}{(x_{2'} - x_1)^\beta (x_2 - x_{1'})^\gamma}, \quad (2.12)$$

$$\alpha = u - \ell_1 - \ell_2 + 1, \quad \beta = u + \ell_1 - \ell_2 + 1, \quad \gamma = u - \ell_1 + \ell_2 + 1, \quad \delta = u + \ell_1 + \ell_2 - 1.$$

The integral representation of the Euler Beta function involving the closed double-loop Pochhammer contour used for formulating its analytic continuation shows how the contours in (2.11) are to be chosen.

3 The deformed algebra

We would like to mention the two relations out of the underlying algebraic structure which will be important in the following. We use the notation $[a] = \frac{q^a - q^{-a}}{q - q^{-1}}$. Here we suppose that the deformation parameter takes generic values, because new features appear in the case $|q| = 1$ and especially if q is a root of unity.

(1) The deformed commutation relations are

$$[S^0, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = \frac{q^{2S^0} - q^{-2S^0}}{q - q^{-1}} = [2S^0], \quad (3.1)$$

and the corresponding Casimir element is

$$C = S^+ S^- + [S^0][S^0 - 1] = S^- S^+ + [S^0][S^0 + 1]. \quad (3.2)$$

(2) In order to preserve these deformed relations in the action on tensor products $V_1 \otimes V_2$ the operators S_1^a and S_2^a acting on V_1 and V_2 have to be combined in the deformed co-product. There are two versions of the co-product transforming into each other by the replacement $q \leftrightarrow q^{-1}$,

$$\Delta(S^a) = S_{12}^a \quad \text{and} \quad \bar{\Delta}(S^a) = \bar{S}_{12}^a,$$

where

$$S_{12}^+ = S_1^+ q^{S_2^0} + q^{-S_1^0} S_2^+, \quad S_{12}^0 = S_1^0 + S_2^0, \quad S_{12}^- = S_1^- q^{S_2^0} + q^{-S_1^0} S_2^-, \quad (3.3)$$

Representations can be given by defining the action of the above generators on polynomials of the variable x as

$$\hat{S}^{(\ell)-} = \hat{S}^{(0)-}, \quad \hat{S}^{(\ell)0} = x^{-\ell} \hat{S}^{(0)0} x^\ell, \quad \hat{S}^{(\ell)+} = x^{-2\ell} \hat{S}^{(0)+} x^{2\ell}, \quad (3.4)$$

where

$$S^{(0)-} = D_q = \frac{1}{x}[x\partial], \quad S^{(0)0} = x\partial, \quad S^{(0)+} = -x^2 D_q. \quad (3.5)$$

The monomials x^m span a basis and they are eigenvectors of $S^{(\ell)0}$ with eigenvalues $\ell + m$ and eigenvectors of the Casimir operator (3.2) with eigenvalue $[\ell][\ell - 1]$.

The tensor product $V^{(\ell_1)} \otimes V^{(\ell_2)}$ is represented again by polynomials in x_1 and x_2 . The action of the algebra and the decomposition of the product representation into irreducible ones is determined by the co-product (3.3). The subspaces, invariant under S_{12}^a are spanned by $\varphi_n^{(m)}(x_1, x_2|q) = (S_{12}^+)^m \cdot \varphi_n(x_1, x_2|q)$. Here n labels the invariant subspace and m labels the basis spanning this subspace. The lowest weight states, obeying

$$S_{12}^{(0)} \varphi_n(x_1, x_2; q) = (\ell_1 + \ell_2 + n) \varphi_n(x_1, x_2; q), \quad S_{12}^- \varphi_n(x_1, x_2; q) = 0, \quad (3.6)$$

are given explicitly by

$$\varphi_n(x_1, x_2|q) = \varphi_n(x_1, x_2; \ell_1, \ell_2; q) = \prod_{k=1}^n (q^{-\ell_1+1-k} x_1 - q^{k-1+\ell_2} x_2) \quad (3.7)$$

We shall suppress usually the arguments ℓ_1, ℓ_2 if it is clear to which representations the variables x_1, x_2 refer.

The corresponding Casimir operator has the form

$$C_{12} = q^{S_2^0 - S_1^0 + 1} S_1^+ S_2^- + q^{S_2^0 - S_1^0 - 1} S_1^- S_2^+ - (q - q^{-1})^{-2} \left((q + q^{-1}) \left(1 + q^{2S_2^0 - 2S_1^0} \right) + q^{2S_2^0} [\ell_1][\ell_1 - 1] + q^{-2S_1^0} [\ell_2][\ell_2 - 1] \right) \quad (3.8)$$

and we have the eigenvalue relation

$$\hat{C}_{12} \varphi_n^{(m)}(x_1, x_2|q) = [n + \ell_1 + \ell_2][n + \ell_1 + \ell_2 - 1] \varphi_n^{(m)}(x_1, x_2|q).$$

For the co-product $\bar{\Delta}$ the basis of the invariant subspaces is given by

$$\bar{\varphi}_n^{(m)}(x_1, x_2|q) = \varphi_n^{(m)}(x_1, x_2|q^{-1})$$

We have the analogous relations for the co-product operators $\bar{S}_{12}^a, \bar{C}_{12}$ where in all explicit expressions the replacement $q \leftrightarrow q^{-1}$ has to be done.

The definition of the co-product allows an extension involving an additional parameter u .

$$\Delta_u(S^a) = S_{12,u}^a,$$

$$S_{12,u}^\pm = q^{\mp \frac{u}{2} + S_1^0} S_1^\pm + q^{\pm \frac{u}{2} - S_1^0} S_2^\pm, \quad S_{12,u}^0 = S_1^0 + S_2^0. \quad (3.9)$$

Now the basis of invariant subspaces is spanned by $\varphi_n^{(m)}(x_1, x_2|q, u)$, where

$$\varphi_n^{(m)}(x_1, x_2|q, u) = (S_{12,u}^+)^m \varphi_n(x_1, x_2|q, u)$$

$$\varphi_n(x_1, x_2|q, u) = \varphi_n(q^{-\frac{u}{2}} x_1, q^{\frac{u}{2}} x_2|q) = \varphi_n(x_1, x_2; \ell_1 + \frac{u}{2}, \ell_2 + \frac{u}{2}; q). \quad (3.10)$$

The Casimir operator (3.2) is represented as

$$C_{12,u} = q^{S_2^0 - S_1^0} (q^{2u-1} S_1^- S_2^+ + q^{1-2u} S_1^+ S_2^-) + \frac{q + q^{-1}}{(q - q^{-1})^2} (q^{2S_2^0} + q^{-2S_1^0} - q^{2S_2^0 - 2S_1^0} - 1) + q^{2S_2^0} [\ell_1][\ell_1 - 1] + q^{2S_1^0} [\ell_2][\ell_2 - 1] \quad (3.11)$$

$\varphi_n^{(m)}(x_1, x_2|q, u)$ are its eigenfunctions with eigenvalues $[\ell_1 + \ell_2 + n][\ell_1 + \ell_2 + n - 1]$.

Correspondingly, the co-product $\bar{\Delta}_u$ and the basis polynomials $\bar{\varphi}_n^{(m)}(x_1, x_2|q, u)$ are defined by the relations obtained from the above ones by the replacement $q \leftrightarrow q^{-1}$. This simple remark turns out to be essential in the following.

The functions $\varphi_n(x_1, x_2; \ell_1, \ell_2; q)$ represent the lowest weight states and in this sense they are the appropriate deformations of $\phi_n(x_1, x_2) = (x_1 - x_2)^n$. The way of deformation depends on ℓ_1, ℓ_2 and on the co-product.

The conditions from which we have determined the wave functions representing $V^{(\ell)}$ or $V^{(\ell_1)} \otimes V^{(\ell_2)}$ are difference equations, more precisely difference equations in $t_i = \ln x_i / \ln q$. In particular besides of the constant function 1 any function periodic in t with period 1, i.e. with the property $f(q^N x) = f(x)$, N integer, could represent the lowest weight state of $V^{(\ell)}$. The restriction to polynomial dependence on x_i fixes the wave functions uniquely.

In the undeformed case the generalization of ϕ_n to non-integer powers α plays an important role in particular in constructing the integral kernel of the R operator. Therefore we discuss the corresponding generalization of the deformed function φ_n .

For the continuation of φ_n (3.7) away from integer values n the latter variable should be released from its role of a limit of the index range in the product. Therefore we write

$$\begin{aligned}\varphi_n(x_1, x_2; \ell_1, \ell_2; q) &= x_1^n q^{-n\ell_1 - \frac{1}{2}n(n-1)} \prod_{k=1}^n \left(1 - \frac{x_2}{x_1} q^{2k-2+\ell_1+\ell_2}\right) = \\ &= (-x_2)^n q^{n\ell_2 + \frac{1}{2}n(n-1)} \prod_{k=1}^n \left(1 - \frac{x_1}{x_2} q^{-2k+2-\ell_1-\ell_2}\right),\end{aligned}\quad (3.12)$$

and change the index k to $k' = n - k + 1$,

$$\prod_{k=1}^n \left(1 - \frac{x_1}{x_2} q^{2n-2k+\ell_1+\ell_2}\right) = \prod_1^\infty \frac{\left(1 - \frac{x_1}{x_2} q^{2n-2k+\ell_1+\ell_2}\right)}{\left(1 - \frac{x_1}{x_2} q^{-2k+\ell_1+\ell_2}\right)}.\quad (3.13)$$

and similar for the other form. The continuation to arbitrary "powers" α is given by

$$\varphi_\alpha(x_1, x_2, \ell_1, \ell_2; q) = \begin{cases} x_1^\alpha q^{-\alpha\ell_1 - \frac{1}{2}\alpha(\alpha-1)} g\left(\frac{x_2}{x_1} q^{\ell_1+\ell_2}, \alpha, q^{-2}\right), & |q| > 1, \\ (-x_2)^\alpha q^{\alpha\ell_2 + \frac{1}{2}\alpha(\alpha-1)} g\left(\frac{x_1}{x_2} q^{-\ell_1-\ell_2}, \alpha, q^2\right), & |q| < 1. \end{cases}\quad (3.14)$$

where we define for $|q| < 1$ (compare [35])

$$g(z, a, q) = \prod_1^\infty \left(\frac{1 - zq^{-a+k}}{1 - zq^k} \right) = \frac{(zq^{a+1}, q)_\infty}{(zq, q)_\infty}.\quad (3.15)$$

φ_α is homogeneous of degree α in the variables x_1, x_2 . For dilatations by factors of q or q^{-1} we have further homogeneity properties,

$$\begin{aligned}\varphi_\alpha(qx_1, q^{-1}x_2; \ell_1, \ell_2; q) &= \left(\frac{x_1 q^{-\ell_1+1} - x_2 q^{\ell_2-1}}{x_1 q^{-\ell_1+1-\alpha} - x_2 q^{\ell_2-1+\alpha}} \right) \varphi_\alpha(x_1, x_2; \ell_1, \ell_2; q), \\ \varphi_\alpha(q^{-1}x_1, qx_2; \ell_1, \ell_2; q) &= \left(\frac{x_1 q^{-\ell_1-\alpha} - x_2 q^{\ell_2+\alpha}}{x_1 q^{-\ell_1} - x_2 q^{\ell_2}} \right) \varphi_\alpha(x_1, x_2; \ell_1, \ell_2; q).\end{aligned}\quad (3.16)$$

The multiplication rule holds,

$$\varphi_\alpha(x_1, x_2; \ell_1, \ell_2; q) \varphi_\beta(x_1, x_2; \ell_1 + \alpha, \ell_2 + \alpha; q) = \varphi_{\alpha+\beta}(x_1, x_2; \ell_1, \ell_2; q),\quad (3.17)$$

related to the the corresponding property of the undeformed $\phi_\alpha = (x_1 - x_2)^\alpha$. For $\beta = -\alpha$ it results in the inversion rule,

$$(\varphi_\alpha(x_1, x_2; \ell_1, \ell_2; q))^{-1} = \varphi_{-\alpha}(x_1, x_2; \ell_1 + \alpha, \ell_2 + \alpha; q).\quad (3.18)$$

The relations are easily seen to hold for integer $\alpha = n$ (3.7) and to continue to arbitrary values. The following relation for integer values

$$\varphi_n(x_2, x_1; \ell_2, \ell_1; q) = (-1)^n \varphi_n(x_1, x_2; \ell_1, \ell_2; q^{-1}).\quad (3.19)$$

implies the crossing rule

$$\varphi_\alpha(x_2, x_1; \ell_2, \ell_1; q) = C_\alpha \varphi_\alpha(x_1, x_2; \ell_1, \ell_2; q^{-1}),\quad (3.20)$$

where C_α does not depend on x_1, x_2 . We notice that the action of the co-product generators S_{12}^0, S_{12}^- (3.6) continues to arbitrary α as well. Finally we write the action of the finite difference operators,

$$\begin{aligned}D_{q,1}\varphi_\alpha(x_1, x_2; \ell_1, \ell_2; q) &= [\alpha] q^{-\ell_1 + \frac{1}{2} - \frac{\alpha}{2}} \varphi_{\alpha-1}(q^{-\frac{1}{2}}x_1, x_2; \ell_1, \ell_2; q), \\ D_{q,2}\varphi_\alpha(x_1, x_2; \ell_1, \ell_2; q) &= -[\alpha] q^{\ell_2 - \frac{1}{2} + \frac{\alpha}{2}} \varphi_{\alpha-1}(q^{-\frac{1}{2}}x_1, x_2; \ell_1, \ell_2; q).\end{aligned}\quad (3.21)$$

4 The universal R operator

The 4×4 matrix (2.1) with the non-zero elements a, b, c substituted as

$$a = q^{u+1} - q^{-u-1}, b = q^u - q^{-u}, c = q - q^{-1}. \quad (4.1)$$

is another well known solution of YBE on which the anisotropic XXZ Heisenberg chain of spin $\frac{1}{2}$, of higher spins and related integrable models are based. The Lax operator

$$L_q^{(\ell)}(u) = \begin{pmatrix} q^{u+S^0} - q^{-u-S^0} & (q - q^{-1})S^- \\ (q - q^{-1})S^+ & q^{u-S^0} - q^{-u+S^0} \end{pmatrix}. \quad (4.2)$$

involves the generators of the deformed algebra obeying (3.1). It represents the operator $R^{(-\frac{1}{2}, \ell)}(u + \frac{1}{2})$ acting on the tensor product of representation spaces $V^{-\frac{1}{2}} \otimes V^{(\ell)}$ where the first one is spanned by the two-dimensional column vectors and the latter by polynomials in x . The YBE for $\ell_1 = \ell_2 = -\frac{1}{2}$ and general $\ell_3 = \ell$ is equivalent to the deformed commutation relations (3.1). The particular YBE for arbitrary ℓ_1, ℓ_2 and for $\ell_3 = -\frac{1}{2}$ can be formulated as in (2.4) in terms of the Lax operator and the universal R operator $R_{12}^{(\ell_1 \ell_2)}$ and serves as the defining relation for the latter. Explicitely we have (the subscripts (ℓ_1, ℓ_2) and (12) will be suppressed in the following)

$$\begin{aligned} R(u-v) \begin{pmatrix} q^{u+S_1^0} - q^{-u-S_1^0} & (q - q^{-1})S_1^- \\ (q - q^{-1})S_1^+ & q^{u-S_1^0} - q^{-u+S_1^0} \end{pmatrix} \begin{pmatrix} q^{v+S_2^0} - q^{-v-S_2^0} & (q - q^{-1})S_2^- \\ (q - q^{-1})S_2^+ & q^{v-S_2^0} - q^{-v+S_2^0} \end{pmatrix} = \\ \begin{pmatrix} q^{v+S_2^0} - q^{-v-S_2^0} & (q - q^{-1})S_2^- \\ (q - q^{-1})S_2^+ & q^{v-S_2^0} - q^{-v+S_2^0} \end{pmatrix} \begin{pmatrix} q^{u+S_1^0} - q^{-u-S_1^0} & (q - q^{-1})S_1^- \\ (q - q^{-1})S_1^+ & q^{u-S_1^0} - q^{-u+S_1^0} \end{pmatrix} R(u-v), \end{aligned} \quad (4.3)$$

Separating terms depending only on $u - v$ from the ones depending also on $u + v$ we arrive at the following set of equations:

$$[R(u), q^{S_1+S_2}] = 0,$$

$$\begin{aligned} R(u) \left(q^{\frac{u}{2}+S_1} S_2^- + q^{-\frac{u}{2}-S_2} S_1^- \right) &= \left(q^{\frac{u}{2}-S_1} S_2^- + q^{-\frac{u}{2}+S_2} S_1^- \right) R(u), \\ R(u) \left(q^{\frac{u}{2}-S_1} S_2^+ + q^{-\frac{u}{2}+S_2} S_1^+ \right) &= \left(q^{\frac{u}{2}+S_1} S_2^+ + q^{-\frac{u}{2}-S_2} S_1^+ \right) R(u), \\ R(u) \left(q^{\frac{u}{2}+S_2} S_1^- + q^{-\frac{u}{2}-S_1} S_2^- \right) &= \left(q^{\frac{u}{2}-S_2} S_1^- + q^{-\frac{u}{2}+S_1} S_2^- \right) R(u), \\ R(u) \left(q^{\frac{u}{2}-S_2} S_1^+ + q^{-\frac{u}{2}+S_1} S_2^+ \right) &= \left(q^{\frac{u}{2}+S_2} S_1^+ + q^{-\frac{u}{2}-S_1} S_2^+ \right) R(u), \end{aligned} \quad (4.4)$$

$$\begin{aligned} R(u) \left(q^{u+S_1-S_2} + q^{-u+S_2-S_1} - (q - q^{-1})^2 S_1^- S_2^+ \right) &= \\ \left(q^{u+S_1-S_2} + q^{-u+S_2-S_1} - (q - q^{-1})^2 S_1^+ S_2^- \right) R(u), \\ R(u) \left(q^{u+S_2-S_1} + q^{-u+S_1-S_2} - (q - q^{-1})^2 S_1^+ S_2^- \right) &= \\ \left(q^{u+S_2-S_1} + q^{-u+S_1-S_2} - (q - q^{-1})^2 S_1^- S_2^+ \right) R(u). \end{aligned} \quad (4.5)$$

It is remarkable that these relations can be written in terms of the u -dependent deformed generators $S_{12, \pm u}^a$ and $\bar{S}_{12, \pm u}^a$ and their corresponding Casimir operators.

$$\begin{aligned} R(u) \bar{S}_u^a &= S_{-u}^a R(u), \\ R(u) S_u^a &= \bar{S}_{-u}^a R(u), \end{aligned}$$

$$\hat{C}_{-u} R(u) = R(u) \hat{C}_u, \quad \hat{\bar{C}}_{-u} R(u) = R(u) \hat{\bar{C}}_u, \quad (4.6)$$

In terms of the co-product symbols (3.3) this can be summarized as

$$R(u) \Delta_u = \bar{\Delta}_{-u} R(u), \quad R(u) \bar{\Delta}_u = \Delta_{-u} R(u), \quad (4.7)$$

expressing the basic intertwining property of the R operator [10]. In this way the meaning of the deformed $sl(2)$ symmetry of the R operator becomes evident: The different actions on and irreducible decompositions of the tensor product $V^{(\ell_1)} \otimes V^{(\ell_2)}$ defined according to the different u -dependent co-products Δ_u and $\bar{\Delta}_u$ are mapped by $R(u)$ isomorphically with respect to the algebra (3.1) to the ones defined by $\bar{\Delta}_{-u}$ and Δ_{-u} .

We do have explicitly the irreducible decompositions of $V^{(\ell_1)} \otimes V^{(\ell_2)}$ with respect to $\Delta_{\pm u}$ and $\bar{\Delta}_{\pm u}$ in terms of the basis polynomials $\varphi_n^{(m)}(x_1, x_2|q, u)$ and $\bar{\varphi}_n^{(m)}(x_1, x_2|q, u)$. The intertwining property implies the eigenvalue relations

$$\begin{aligned} R(u) \varphi_n^{(m)}(x_1, x_2|q, u) &= \bar{R}_n \bar{\varphi}_n^{(m)}(x_1, x_2|q, -u) \\ R(u) \bar{\varphi}_n^{(m)}(x_1, x_2|q, u) &= R_n \varphi_n^{(m)}(x_1, x_2|q, -u) \end{aligned} \quad (4.8)$$

The general relation (4.7) implies also that applying these eigenvalue relations to one projection (e.g. $a \hat{=}$ $-$) of the defining relations (4.6)) is enough to define the universal R operator in terms of the eigenvalues $R_n(u), \bar{R}_n(u)$.

To formulate this explicitly from the perspective of section 2 let us call the first set of 3 relations in (4.6) symmetry relations and call the transformations S_{-u}^a left symmetry and \bar{S}_u^a right symmetry of $R(u)$. The remaining relations can be formulated as (compare (2.8))

$$R(u) \begin{pmatrix} K^{+-}(u) & K^-(u) \\ K^+(u) & K^{-+}(u) \end{pmatrix} = \begin{pmatrix} \bar{K}^{+-}(u) & \bar{K}^-(u) \\ \bar{K}^+(u) & \bar{K}^{-+}(u) \end{pmatrix} R(u) \quad (4.9)$$

where

$$\begin{aligned} K^- &= S_u^-, \quad \bar{K}^- = \bar{S}_{-u}^-, \quad K^+ = S_u^+, \quad \bar{K}^+ = \bar{S}_{-u}^+, \\ K^{+-} &= q^{u+S_1-S_2} + q^{-u+S_2-S_1} - (q - q^{-1})^2 S_1^- S_2^+, \\ \bar{K}^{+-} &= q^{u+S_1-S_2} + q^{-u+S_2-S_1} - (q - q^{-1})^2 S_1^+ S_2^-, \\ K^{-+} &= q^{u+S_2-S_1} + q^{-u+S_1-S_2} - (q - q^{-1})^2 S_1^+ S_2^-, \\ \bar{K}^{-+} &= q^{u+S_2-S_1} + q^{-u+S_1-S_2} - (q - q^{-1})^2 S_1^- S_2^+. \end{aligned}$$

The mutual dependence of the relations due to the deformed symmetry is established by checking the following relations of covariance by the right symmetry \bar{S}_u^a for K^a and by the left symmetry S_{-u}^a for \bar{K}^a ,

$$\begin{aligned} [\bar{S}_u^-, K^-] &= 0 = [S_{-u}^-, \bar{K}^-], \quad [\bar{S}_u^+, K^+] = 0 = [S_{-u}^+, \bar{K}^+], \\ [\bar{S}_u^-, K^+] &= (q - q^{-1})^{-1} (q^{-S_1-S_2} K^{+-} - q^{S_1+S_2} K^{-+}), \\ [S_{-u}^-, \bar{K}^+] &= (q - q^{-1})^{-1} (-q^{-S_1-S_2} \bar{K}^{+-} + q^{S_1+S_2} \bar{K}^{-+}), \end{aligned}$$

$$[\bar{S}_u^+, K^-] = (q - q^{-1})^{-1} (-q^{-S_1-S_2} K^{-+} + q^{S_1+S_2} K^{+-}),$$

$$[S_{-u}^-, \bar{K}^+] = (q - q^{-1})^{-1} (q^{-S_1-S_2} \bar{K}^{-+} - q^{S_1+S_2} \bar{K}^{+-}).$$

We would like to calculate the action of the $a^{\hat{=}}$ components of the relations (4.6) to the polynomials representing the lowest weight vectors $\varphi_n(x_1, x_2|q^{\pm 1}, \pm u)$. The operator S_u^- acts on them as

$$S_u^- \begin{pmatrix} \varphi_n(x_1, x_2|q, u) \\ \bar{\varphi}_n(x_1, x_2|q, u) \\ \varphi_n(x_1, x_2|q, -u) \\ \bar{\varphi}_n(x_1, x_2|q, -u) \end{pmatrix} = \begin{pmatrix} 0 \\ (q - q^{-1})[n][u + n - 1 - \ell_1 - \ell_2] \bar{\varphi}_{n-1}(x_1, x_2|q, -u) \\ (q - q^{-1})[u][n] q^{\ell_2 - \ell_1} \varphi_{n-1}(q^{-1}x_1, qx_2|q, -u) \\ (q - q^{-1})[n][n - 1 + \ell_1 + \ell_2] \bar{\varphi}_{n-1}(x_1, x_2|q, -u) \end{pmatrix} \quad (4.10)$$

More relations are obtained by the replacements $q \leftrightarrow q^{-1}$ or $u \leftrightarrow -u$. In fact we use now the first two relations only. The relations hold also if continuing from integer n to arbitrary α .

Applying now the $a^{\hat{=}}$ component of the second relation in (4.6) on $\varphi_n(x_1, x_2|q, u)$ we obtain zero on both sides and applying on $\bar{\varphi}_n(x_1, x_2|q, u)$ we obtain the recurrence relation for the eigenvalues R_n ,

$$R_n = -R_{n-1} \frac{[u + n - 1 + \ell_1 + \ell_2]}{[-u + n - 1 + \ell_1 + \ell_2]} \quad (4.11)$$

Acting in the analogous way with the $a^{\hat{=}}$ component of the first relation in (4.6) we obtain that the same recurrence relation is obeyed by \bar{R}_n . Since $\varphi_0(x_1, x_2|q, u) = \bar{\varphi}_0(x_1, x_2|q, u) = 1$ we have $R_0 = \bar{R}_0$ and therefore

$$R_n = \bar{R}_n = (-1)^n R_0 \prod_{k=1}^n \frac{[u + k - 1 + \ell_1 + \ell_2]}{[-u + k - 1 + \ell_1 + \ell_2]}. \quad (4.12)$$

In different form the spectral decomposition was obtained first in [8].

5 The integral kernel of the R operator

We would like to represent the R operator acting of polynomial functions $\psi(x_1, x_2)$ in integral form (2.11). The integration is along closed contours in both $x_{1'}$ and $x_{2'}$ in order to allow partial integrations without boundary terms. The defining conditions (4.4, 4.5, 4.6) of the form

$$Q R(u) = R(u) \tilde{Q}, \quad (5.1)$$

where Q and \tilde{Q} are composed out of S_1^a and S_2^a , result in conditions on the kernel of the form

$$(Q_x - \tilde{Q}_{x'}^T) \mathcal{R}(x_1, x_2|x_{1'}, x_{2'}) = 0. \quad (5.2)$$

Q_x acts on x_1, x_2 and $\tilde{Q}_{x'}$ on $x_{1'}$ and $x_{2'}$. \tilde{Q}^T is obtained from \tilde{Q} by partial integration. This conjugation acts on the generators (3.4, 3.5) as

$$(S_1^{a(\ell)})^T = -S_1^{a(1-\ell)}, \quad (S_2^{a(\ell)})^T = -S_2^{a(1-\ell)}. \quad (5.3)$$

We introduce the temporary notations

$$\begin{aligned}
A_1 &= q^{S_1^0+S_2^0} \mathcal{R}(x_1, x_2|x_{1'}, x_{2'}), & A_2 &= q^{S_1^0-S_2^0} \mathcal{R}(x_1, x_2|x_{1'}, x_{2'}), \\
A_3 &= q^{-S_1^0+S_2^0} \mathcal{R}(x_1, x_2|x_{1'}, x_{2'}), & A_4 &= q^{-S_1^0-S_2^0} \mathcal{R}(x_1, x_2|x_{1'}, x_{2'}), \\
B_1 &= q^{S_{1'}^0+S_{2'}^0} \mathcal{R}(x_1, x_2|x_{1'}, x_{2'}), & B_2 &= q^{S_{1'}^0-S_{2'}^0} \mathcal{R}(x_1, x_2|x_{1'}, x_{2'}), \\
B_3 &= q^{-S_{1'}^0+S_{2'}^0} \mathcal{R}(x_1, x_2|x_{1'}, x_{2'}), & B_4 &= q^{-S_{1'}^0-S_{2'}^0} \mathcal{R}(x_1, x_2|x_{1'}, x_{2'}).
\end{aligned} \tag{5.4}$$

The lower subscripts on the generators indicate that they act on the variable with the corresponding subscript and that they are taken in the corresponding representation $\ell_1, \ell_2, \ell_{1'} = 1 - \ell_1, \ell_{2'} = 1 - \ell_2$. The defining conditions (4.4 4.5 4.6) now appear as 8 linear equations for the unknowns $A_1, \dots, A_4, B_1, \dots, B_4$. The first relation in (4.4) results in

$$A_1 - B_4 = 0, \quad A_4 - B_1 = 0. \tag{5.5}$$

The next two equations in (4.4) appear as

$$\begin{aligned}
&\frac{q^{-\frac{u}{2}-\ell_1}}{x_1} A_1 + \left(\frac{q^{\frac{u}{2}-\ell_2}}{x_2} - \frac{q^{-\frac{u}{2}+\ell_1}}{x_1} \right) A_3 - \frac{q^{\frac{u}{2}+\ell_2}}{x_2} A_4 \\
&\quad + ((1, 2) \leftrightarrow (1', 2'), A_i \leftrightarrow B_i) = 0, \\
&-q^{\frac{u}{2}+\ell_1} x_1 A_1 + (q^{\frac{u}{2}-\ell_1} x_1 - q^{-\frac{u}{2}+\ell_2} x_2) A_3 + q^{-\frac{u}{2}-\ell_2} x_2 A_4 \\
&\quad + ((1, 2) \leftrightarrow (1', 2'), A_i \leftrightarrow B_i) = 0.
\end{aligned} \tag{5.6}$$

The second line in these equations is obtained from the explicitly written first line by substituting the subscripts on x and ℓ as $1 \leftrightarrow 1', 2 \leftrightarrow 2'$ and by replacing A_i by B_i . The equations resulting from the next two in (4.4) have a similar form and can be obtained from (5.6) by the replacement of the lower subscripts on x and ℓ as $1 \leftrightarrow 2, 1' \leftrightarrow 2'$ and by the replacement of the unknowns as $A_3 \rightarrow A_2$ and $B_1 \rightarrow B_2$.

From (4.5) we obtain

$$\begin{aligned}
&-\frac{q^{\frac{u}{2}+\ell_2}}{x_2} (q^{-\frac{u}{2}+\ell_1} x_1 - q^{\frac{u}{2}-\ell_2} x_2) A_2 + \frac{q^{-\frac{u}{2}-\ell_2}}{x_2} (q^{-\frac{u}{2}+\ell_2} x_2 - q^{\frac{u}{2}-\ell_1} x_1) A_3 \\
&\quad + \frac{x_1}{x_2} (q^{\ell_1-\ell_2} A_1 + q^{\ell_2-\ell_1} A_4) \\
&-\frac{q^{-\frac{u}{2}+\ell_{1'}}}{x_{1'}} (q^{-\frac{u}{2}+\ell_{1'}} x_{1'} - q^{\frac{u}{2}-\ell_{2'}} x_{2'}) B_2 + \frac{q^{\frac{u}{2}+\ell_{1'}}}{x_{1'}} (q^{-\frac{u}{2}+\ell_{2'}} x_{2'} - q^{\frac{u}{2}-\ell_{1'}} x_{1'}) B_3 \\
&\quad - \frac{x_{2'}}{x_{1'}} (q^{\ell_{2'}-\ell_{1'}} B_1 + q^{\ell_{1'}-\ell_{2'}} B_4) = 0.
\end{aligned} \tag{5.7}$$

and the analogous equation obtained from the latter by the substitution of subscripts on x and ℓ as $1 \leftrightarrow 2, 1' \leftrightarrow 2'$.

We start solving this linear system by excluding B_i and arrive at

$$\begin{aligned}
A_2 &= \frac{(q^{-\frac{u}{2}+\ell_1-1} x_{1'} - q^{\frac{u}{2}+\ell_2} x_2) (q^{-\frac{u}{2}+\ell_1-1} x_2 - q^{\frac{u}{2}-\ell_2+1} x_1)}{(q^{-\frac{u}{2}+\ell_1} x_1 - q^{\frac{u}{2}-\ell_2} x_2) (q^{-\frac{u}{2}+\ell_1-1} x_2 - q^{\frac{u}{2}+\ell_2} x_{1'})} A_1, \\
A_3 &= \frac{(q^{-\frac{u}{2}+\ell_2-1} x_1 - q^{\frac{u}{2}-\ell_1+1} x_2) (q^{-\frac{u}{2}+\ell_2-1} x_{2'} - q^{\frac{u}{2}+\ell_1} x_1)}{(q^{-\frac{u}{2}+\ell_2} x_2 - q^{\frac{u}{2}-\ell_1} x_1) (q^{-\frac{u}{2}+\ell_2-1} x_1 - q^{\frac{u}{2}+\ell_1} x_{2'})} A_1, \\
A_4 &= \frac{(q^{-\frac{u}{2}+\ell_1-1} x_{1'} - q^{\frac{u}{2}+\ell_2} x_2) (q^{-\frac{u}{2}-\ell_1} x_{2'} - q^{\frac{u}{2}+\ell_2+1} x_1)}{(q^{-\frac{u}{2}-\ell_2} x_2 - q^{-\frac{u}{2}-\ell_1+1} x_{1'}) (q^{-\frac{u}{2}+\ell_2-1} x_1 - q^{\frac{u}{2}+\ell_1} x_{2'})} A_1.
\end{aligned} \tag{5.8}$$

B_1, B_4 are known by (5.5). B_2, B_3 are obtained by excluding A_i ,

$$\begin{aligned} B_2 &= \frac{(q^{-\frac{u}{2}+\ell_1-1}x_{1'} - q^{\frac{u}{2}+\ell_2}x_2)(q^{-\frac{u}{2}-\ell_1}x_{2'} - q^{\frac{u}{2}+\ell_2}x_{1'})}{(q^{-\frac{u}{2}+\ell_1-1}x_2 - q^{\frac{u}{2}+\ell_2}x_{1'})(q^{-\frac{u}{2}-\ell_1+1}x_{1'} - q^{\frac{u}{2}+\ell_2-1}x_{2'})} A_1, \\ B_3 &= \frac{(q^{-\frac{u}{2}-\ell_2}x_{1'} - q^{\frac{u}{2}+\ell_1}x_{2'})(q^{-\frac{u}{2}+\ell_2-1}x_{2'} - q^{\frac{u}{2}+\ell_1}x_1)}{(q^{-\frac{u}{2}-\ell_2+1}x_{2'} - q^{\frac{u}{2}+\ell_1-1}x_{1'})(q^{-\frac{u}{2}+\ell_2-1}x_1 - q^{\frac{u}{2}+\ell_1}x_{2'})} A_1. \end{aligned} \quad (5.9)$$

We substitute A_i, B_i by the definitions in terms of \mathcal{R} (5.4) and obtain that the defining conditions result in four difference equations,

$$\begin{aligned} \mathcal{R}(q^2x_1, x_2|x_{1'}, x_{2'}) &= q^{-2\ell_1} \frac{(q^{-\frac{u}{2}+\ell_2-1}x_2 - q^{\frac{u}{2}-\ell_1+1}x_1)(q^{-\frac{u}{2}+\ell_2}x_1 - q^{\frac{u}{2}+\ell_1}x_{2'})}{(q^{-\frac{u}{2}+\ell_2}x_1 - q^{\frac{u}{2}-\ell_1}x_2)(q^{-\frac{u}{2}+\ell_2-1}x_{2'} - q^{\frac{u}{2}+\ell_1+1}x_1)} \mathcal{R}(x_1, x_2|x_{1'}, x_{2'}), \\ \mathcal{R}(x_1, q^2x_2|x_{1'}, x_{2'}) &= q^{-2\ell_2} \frac{(q^{-\frac{u}{2}+\ell_1-1}x_1 - q^{\frac{u}{2}-\ell_2+1}x_2)(q^{-\frac{u}{2}+\ell_1}x_2 - q^{\frac{u}{2}+\ell_2}x_{1'})}{(q^{-\frac{u}{2}+\ell_1-1}x_{1'} - q^{\frac{u}{2}+\ell_2+1}x_2)(q^{-\frac{u}{2}+\ell_1}x_2 - q^{\frac{u}{2}-\ell_2}x_1)} \mathcal{R}(x_i), \\ \mathcal{R}(x_1, x_2|q^2x_{1'}, x_{2'}) &= \frac{q^{-2+2\ell_1}(q^{-\frac{u}{2}+\ell_1}x_{1'} - q^{\frac{u}{2}+\ell_2}x_2)(q^{-\frac{u}{2}-\ell_1}x_{2'} - q^{\frac{u}{2}+\ell_2}x_{1'})}{(q^{-\frac{u}{2}+\ell_1-1}x_2 - q^{\frac{u}{2}+\ell_2+1}x_{1'})(q^{-\frac{u}{2}-\ell_1+1}x_{1'} - q^{\frac{u}{2}+\ell_2-1}x_{2'})} \mathcal{R}(x_i), \\ \mathcal{R}(x_1, x_2|x_{1'}, q^2x_{2'}) &= \frac{q^{-2+2\ell_2}(q^{-\frac{u}{2}+\ell_2}x_{2'} - q^{\frac{u}{2}+\ell_1}x_1)(q^{-\frac{u}{2}-\ell_2}x_{1'} - q^{\frac{u}{2}+\ell_1}x_{2'})}{(q^{-\frac{u}{2}+\ell_2-1}x_1 - q^{\frac{u}{2}+\ell_1+1}x_{2'})(q^{-\frac{u}{2}-\ell_2+1}x_{2'} - q^{\frac{u}{2}+\ell_1-1}x_{1'})} \mathcal{R}(x_i) \end{aligned} \quad (5.10)$$

Comparing the dependence on $x_1, x_2, x_{1'}, x_{2'}$ of the factors appearing on the r.h.s in (5.10) with (3.16) leads to the ansatz

$$\mathcal{R}(x_1, x_2|x_{1'}, x_{2'}) = \frac{\varphi_\alpha(x_1, x_2; a_1, a_2; q) \varphi_\delta(x_{1'}, x_{2'}; d_{1'}, d_{2'}; q)}{\varphi_\beta(x_{2'}, x_1; b_{2'}, b_1; q) \varphi_\gamma(x_2, x_{1'}; c_2, c_{1'}; q)} \mathcal{R}_0. \quad (5.11)$$

This is indeed a deformation of the kernel found for the undeformed case(2.12). The detailed comparison of the ansatz with (5.10) results in the following conditions for the parameters,

$$\begin{aligned} \alpha &= u - \ell_1 - \ell_2 + 1, \quad a_1 = a - \frac{u}{2} + \ell_1, \quad a_2 = -a - \frac{u}{2} + \ell_2, \\ \beta &= u + \ell_1 - \ell_2 + 1, \quad b_{2'} = b - \frac{u}{2} - \ell_1, \quad b_1 = -b - \frac{u}{2} + \ell_2, \\ \gamma &= u - \ell_1 + \ell_2 + 1, \quad c_2 = c - \frac{u}{2} - \ell_2, \quad c_{1'} = -c - \frac{u}{2} + \ell_1, \\ \delta &= u + \ell_1 + \ell_2 - 1, \quad d_{1'} = d - \frac{u}{2} - \ell_2 + 1, \quad d_{2'} = -d - \frac{u}{2} - \ell_1 + 1. \end{aligned} \quad (5.12)$$

a, b, c, d remain unconstrained; they affect the normalization only. Notice that each equation of (5.10) fixes the parameter triple of two functions in the ansatz. The underlying YBE guarantees the compatibility of the four difference equations (5.10) and in particular the compatibility of the conditions on the parameters in our ansatz.

The kernel of the universal R operator is given by (5.11) with the parameters substituted as in (5.12).

Being the solution of the difference equations (5.10) the integral kernel (5.11) is not unique. Expressions differing from the given one by factors being periodic functions of $x_1, x_2, x_{1'}, x_{2'}$ (in the multiplicative sense, $f(q^N x) = f(x)$, N integer) solve (5.10) as well. (5.11) is the solution close to the undeformed kernel in its analytic properties.

6 Deformed Beta integrals

Consider the functions $\varphi_\alpha(x_1, x_2; \ell_1, \ell_2; q)$ for particular values $\ell_1 = \ell_2 = \frac{1-\alpha}{2}$ and define

$$\pi_\alpha(x_1, x_2) = \varphi_\alpha(x_1, x_2; \frac{1-\alpha}{2}, \frac{1-\alpha}{2}; q) = \varphi_\alpha(x_1, x_2; \frac{1-\alpha}{2}, \frac{1-\alpha}{2}; q^{-1}). \quad (6.1)$$

The lowest weight property (3.6) implies in this case

$$\left(S_1^{(\frac{-\alpha}{2}),a} + S_2^{(\frac{-\alpha}{2}),a} \right) \pi_\alpha(x_1, x_2) = 0. \quad (6.2)$$

This means that $\pi_\alpha(x_1, x_2)$ are the appropriate deformation of the conformal two-point functions.

(6.2) is checked by writing (3.6) explicitly for the case $\ell_1 = \ell_2 = -\frac{\alpha}{2}$. The check goes straightforward for the component $a = 0$; explicitly we have

$$\begin{aligned} \left(S_1^{\ell_1,0} + S_2^{\ell_2,0} - \ell_1 - \ell_2 - \alpha \right) \varphi_\alpha(x_1, x_2, \ell_1, \ell_2; q) = \\ (x_1 \partial_1 + x_2 \partial_2 - \alpha) \varphi_\alpha(x_1, x_2, \ell_1, \ell_2; q) = 0. \end{aligned} \quad (6.3)$$

Then by applying the latter relation $S_{12}^- \varphi_\alpha(x_1, x_2, -\frac{\alpha}{2}, -\frac{\alpha}{2}) = 0$ implies (6.2) for the component $a = -$. Further, (6.3) leads to

$$(S_1^- + S_2^-) \varphi_\alpha(x_1, x_2, \ell_1, \ell_2; q) = (x_1 x_2)^{-1} \left(S_1^{-\frac{\alpha}{2},+} + S_2^{-\frac{\alpha}{2},+} \right) \varphi_\alpha(x_1, x_2, \ell_1, \ell_2; q) \quad (6.4)$$

and therefore the relation (6.2) for the components $a = 0$ and $a = -$ imply the relation for $a = +$.

We show that

$$\int_C \pi_{\alpha-1}(x_1, x') \pi_{\beta-1}(x', x_2) dx' = \pi_{\alpha+\beta-1}(x_1, x_2) B_C(\alpha, \beta; q). \quad (6.5)$$

The closed contour C should be chosen such that the integral exists and is not vanishing identically. The factor independent of x_1, x_2 on the right-hand side is proportional to the deformed Beta function,

$$B_C(\alpha, \beta; q) = B_C^0 \frac{\Gamma_q(\alpha) \Gamma_q(\beta)}{\Gamma_q(\alpha + \beta)}, \quad \Gamma_q(\alpha + 1) = [\alpha] \Gamma_q(\alpha). \quad (6.6)$$

We check first that the left-hand side of (6.5) is annihilated by the action of $S_1^{(\frac{1-\alpha-\beta}{2}),a} + S_2^{(\frac{1-\alpha-\beta}{2}),a}$. For the component $a = -$ the generators are represented by the finite difference operators for which integration by parts applies and the property of π_α (6.2) implies the corresponding property for the l.h.s. For the component $a = 0$ the assertion is checked easily since the dilatation is not deformed. After this the relations (6.3) and (6.4) imply that also the $a = +$ component annihilates the l.h.s of (6.5). Relying on this we see that the integral has the above form with some factor $B_C(\alpha, \beta; q)$ independent of x_1, x_2 . By acting with the finite difference operators on both sides using (3.21) we obtain iterative relations for $B_C(\alpha, \beta; q)$,

$$\begin{aligned} [\alpha - 1] B_C(\alpha - 1, \beta) &= [\alpha + \beta - 1] B_C(\alpha, \beta), \\ [\beta - 1] B_C(\alpha, \beta - 1) &= [\alpha + \beta - 1] B_C(\alpha, \beta). \end{aligned} \quad (6.7)$$

In this way we confirm (6.6).

(6.5) is the appropriate deformation of the classical Beta integral. The singularities of the integrand are series of poles and branch points of the type $(x-a)^\alpha$, $(x-b)^\beta$. The example of the classical Beta function shows, how closed contours can be chosen (Pochhammer double-loop contour).

Consider now the action of the integral operator (5.11, 5.12) on the lowest weight functions $\varphi_n(x_1, x_2|q, u)$ (3.10). We would like to check by explicit calculations that these functions are eigenfunctions in the sense of (4.8) and that the eigenvalues R_n are the ones obtained previously (4.12).

$$R(u)\varphi_n(x_1, x_2|q, u) = \int_{C_1} dx_{1'} \int_{C_2} dx_{2'} \mathcal{R}(x_1, x_2|x_{1'}, x_{2'}) \varphi_n(x_1, x_2; \frac{u}{2} + \ell_1, \frac{u}{2} + \ell_2; q) \quad (6.8)$$

We substitute the kernel (5.11, 5.12) and observe owing to (3.17, 6.1) that

$$\begin{aligned} & \varphi_\delta(x_{1'}, x_{2'}; d_{1'}, d_{2'}; q) \varphi_n(x_{1'}, x_{2'}; \frac{u}{2} + \ell_1, \frac{u}{2} + \ell_2; q) \\ &= q^{\frac{1}{2}(n-\ell_1+\ell_2)(u+\ell_1+\ell_2+n-1)} \pi_{u+\ell_1+\ell_2+n-1}(q^{-n}x_{1'}, x_{2'}), \\ & (\varphi_\beta(x_{2'}, x_1; b_{2'}, b_1; q))^{-1} = q^{\frac{1}{2}(\ell_1+\ell_2)(-u-\ell_1+\ell_2-1)} \pi_{-u-\ell_1+\ell_2-1}(x_{2'}, x_1), \\ & (\varphi_\gamma(x_2, x_{1'}; c_2, c_{1'}; q))^{-1} = q^{\frac{1}{2}(\ell_1+\ell_2)(-u+\ell_1-\ell_2-1)} \pi_{-u+\ell_1-\ell_2-1}(x_2, x_{1'}), \\ & \varphi_\alpha(x_1, x_2; a_1, a_2; q) = q^{\frac{1}{2}(\ell_1+\ell_2)(u-\ell_1-\ell_2+1)} \pi_{u-\ell_1-\ell_2+1}(x_1, x_2). \end{aligned} \quad (6.9)$$

In (6.8) we have the integral over $x_{2'}$,

$$\begin{aligned} & \int_{C_2} dx_{2'} (\varphi_\beta(x_{2'}, x_1; b_{2'}, b_1; q))^{-1} \varphi_\delta(x_{1'}, x_{2'}; d_{1'}, d_{2'}; q) \varphi_n(x_{1'}, x_{2'}; \frac{u}{2} + \ell_1, \frac{u}{2} + \ell_2; q) = \\ & q^{-\frac{1}{2}n^2 - \frac{n}{2}(-u+2\ell_2-1) - u\ell_1(\ell_1+\ell_2)(\ell_2-\ell_1)-\ell_2} \\ & \pi_{2\ell_2+n-1}(x_{1'}, q^n x_1) B_{C_2}(u + \ell_1 + \ell_2 + n, -u - \ell_1 + \ell_2). \end{aligned} \quad (6.10)$$

Here the first relation in (6.9) and (6.5) have been applied.

Then we have the integral over $x_{1'}$,

$$\begin{aligned} & \int_{C_1} dx_{1'} (\varphi_\gamma(x_2, x_{1'}; c_2, c_{1'}; q))^{-1} \pi_{2\ell_2+n-1}(x_{1'}, q^n x_1) \\ &= q^{\frac{1}{2}(\ell_1+\ell_2)(-u+\ell_1-\ell_2-1)} \pi_{-u+\ell_1+\ell_2+n-1}(x_2, q^n x_1) B_{C_1}(-u + \ell_1 - \ell_2, 2\ell_2 + n). \end{aligned} \quad (6.11)$$

We rewrite the position dependent factor on r.h.s. of (6.11) using the multiplication rule (3.17),

$$\begin{aligned} & \pi_{-u+\ell_1+\ell_2-1}(x_2, q^n x_1) = q^{\frac{n}{2}(-u+\ell_1+\ell_2+n-1)} \\ & \varphi_{-u+\ell_1+\ell_2+n-1}(x_2, x_1; 1 + \frac{n}{2} - \frac{\ell_1+\ell_2}{2}, \frac{n}{2} - \frac{\ell_1+\ell_2}{2}; q) \\ &= q^{\frac{n^2}{2} + \frac{n}{2}(-u+2\ell_2-1)} \pi_{-u+\ell_1+\ell_2-1}(x_2, x_1) \varphi_n(x_2, x_1; -\frac{u}{2} + \ell_2, -\frac{u}{2} + \ell_1; q). \end{aligned} \quad (6.12)$$

In this way we obtain the result of the action of the integral operator (6.8)

$$\begin{aligned} & \int_{C_1} dx_{1'} \int_{C_2} dx_{2'} \mathcal{R}(x_1, x_2|x_{1'}, x_{2'}) \varphi_n(x_1, x_2; \frac{u}{2} + \ell_1, \frac{u}{2} + \ell_2; q) \\ &= q^{-u(2\ell_1+\ell_2)+(\ell_1+\ell_2)\ell_2-\ell_1-2\ell_2} \mathcal{R}_0 \\ & \pi_{u-\ell_1-\ell_2+1}(x_1, x_2) \pi_{-u+\ell_1+\ell_2-1}(x_2, x_1) \varphi_n(x_2, x_1; -\frac{u}{2} + \ell_2, -\frac{u}{2} + \ell_1; q) \end{aligned}$$

$$\begin{aligned}
& B_{C_2}(u + \ell_1 + \ell_2 + n, -u - \ell_1 + \ell_2) B_{C_1}(-u + \ell_1 - \ell_2, 2\ell_2 + n) \\
& = \text{const. } R_n \varphi_n(x_1, x_2; -\frac{u}{2} + \ell_1, -\frac{u}{2} + \ell_2; q^{-1}).
\end{aligned} \tag{6.13}$$

The constant depends on u, ℓ_1, ℓ_2 and of the choice of the contours but not on n . Therefore the normalization \mathcal{R}_0 of the kernel (5.11) can be chosen such that $\text{const.} = 1$. By explicit integration we have shown that the integral form (5.11, 5.12) of the universal R operator obeys the eigenvalue relation (4.8, 4.12).

It may be of interest to check also the Yang-Baxter relation in the integral representation. For this aim one should try to extend the corresponding calculations of [24] and in particular the star-triangle relation for integrals over two-point functions.

7 Discussion

Our study of YBE with deformed $sl(2)$ symmetry follows the scheme used earlier [29, 30]. Generic representations of the algebra of lowest weight on (one-point) functions provide the starting point. As the next step the action on tensor products represented by functions of two points are studied. Then the YBE involving the given Lax operator and the universal R operator is considered as a defining relation of the latter. Evaluating these conditions on the lowest weight two point functions results in the spectral decomposition form of the universal R operator. Writing the conditions with R in integral form results in equations for the kernel. The scheme relies on the well known standard methods [31, 9, 32, 33, 34] reformulated in a way motivated by problems of high energy scattering and in particular adapted to the treatment of non-compact representations.

In our case the action on tensor products is formulated in terms of the generators acting on one-point functions by non-trivial co-products. We introduce the affine extension of the deformed $sl(2)$ co-product involving the spectral parameter; this replaces the explicit treatment of the loop algebra.

We devote much attention to the two-point functions representing the lowest weight states appearing in the irreducible decomposition of the tensor product representations with respect to a particular co-product. They play the key role in formulating the spectral decomposition form of the R operator. Continuing in their expression the integer representation index to arbitrary values we obtain the building blocks for the integral kernel.

Focussing on similarities to the undeformed case we prefer polynomial functions for describing the representations. The eigenvalue equations from which these functions are determined allow other solutions differing from the preferred ones by multiplicatively periodic functions. A formulation in terms of classes of functions where the equivalence relation is given by this periodicity may be reasonable.

The defining relations of the universal R operator resulting from the YBE involving $R(u)$ together with the given Lax operator are shown to be equivalent to the intertwining property of the operator $R(u)$. The intertwining property is the statement that the action of $R(u)$ maps symmetrically the action of the co-products $\Delta_u, \bar{\Delta}_u$ to the ones defined correspondingly by $\bar{\Delta}_{-u}, \Delta_{-u}$. This observation, expected on general grounds [10], is a further key point in our analysis.

We evaluate the intertwining relations on the two-point wave functions representing the lowest weight states in the decomposition of the tensor product with respect to those co-products. We obtain an iterative relation for the eigenvalues which is easily solved. This result provides the R operator in the spectral decomposition form.

Further we study the intertwining relations with the R operator written in integral form. We solve the resulting conditions on the kernel which appear as 4 difference equations. A particular solution for the kernel is obtained in terms of the continued lowest weight two-point functions.

A particular case of the latter functions represent the deformed analogon of the conformal two-point functions. They are used to formulate a deformation of the Beta integral. This is another key point which allows to calculate the action of the integral operator by doing the integrations and to check the integral form of the R operator against its spectral decomposition form.

It is remarkable that these integrations are done relying basically on the symmetry relations. In this way we can avoid the use of extensive summation formulae [35].

It is interesting to compare the calculations step by step with the ones in the undeformed case and to see in particular how the eigenvalues as functions of the representation parameters, the lowest weight and the conformal two-point functions, the defining relations, the kernel of the R operator and the Beta integrals are deformed.

The defining conditions of the undeformed $sl(2)$ symmetric universal R operator are equivalent to a system of 4 first order differential equations. In the deformed case they are replaced by 4 difference equations.

In the undeformed case the defining conditions on the universal R operator decompose in the symmetry condition and a further one. Operators obeying the symmetry condition only act symmetrically on the tensor product; their kernels are conformal 4-point functions. The R operator is a particular symmetric operator. The further condition is to fix the arbitrariness left in the conformal 4-point functions, the dependence on the anharmonic ratio. Under the deformation the symmetry condition generalizes in two independent ways into the intertwining relations. They are equivalent to all defining conditions and fix the R operator completely.

Our results apply to the case of generic values of q . The special features arising for q being a root of unity have not been considered here. Also we did not investigate the interesting behaviour of the above results in the vicinity of such special values of q . Nevertheless, a similarity between the form of the R operator for the root of unity case obtained in [16] and of our integral kernel \mathcal{R} can be observed: both are products of four factors. Whereas in [16] the factors are Boltzmann weights of the chiral Potts model here they are two-point functions, continuations of the functions representing the lowest weight states.

Acknowledgements

One of us (DK) is grateful for hospitality at Leipzig University. The work of two of us (DK and RK) has been supported by the German Federal Ministry BMBF/IB. One of us (MM) has been supported by VW Stiftung.

References.

- [1] R. J. Baxter 1982 *Exactly Solved Models in Statistical Mechanics* (Orland, FL: Academic)
- [2] H. Bethe Z. Phys. 71 (1932) 205.

- [3] L. D. Faddeev, E.K. Sklyanin and L. A. Takhtajan Theor. Math. Phys. 40 (1979) 194. L.A. Takhtajan and L. D. Faddeev, Russ. Math. Surv. 34 (1979)11.
- [4] C.N. Yang and C.P. Yang, Phys. Rev. 150 (1966)321, 327.
- [5] L.A. Takhtajan Phys.Lett. A 87 (1982) 479. H.M. Babujian, Nucl.Phys. B 215 (1983) 317.
- [6] P. Kulish, N. Reshetikhin and E. Sklyanin, Lett. Math Phys. 5 (1981) 393-403.
- [7] A.N. Kirillov, N.Yu. Reshetikhin, J Phys. A: Math. Gen. 20 (1987), 1565-1585; and ibid. 1587-1597.
- [8] M. Jimbo, Lett. Math. Phys. 10 (1985) 63-69.
- [9] P.D. Kulish and N. Yu. Reshetikhin, Zap. Nauch. Semin. LOMI 101 (1981) 101, translation in J. Sov. Math. 23 (1983) 2435.
- [10] V.G. Drinfeld, Quantum groups in: Proc. Intern Congress of Mathematics, Berkeley 1986, I, pp. 798-820, California Acad. Press 1987.
- [11] L.D. Faddeev, N.Y. Reshetikhin and L.A. Takhtadjan, in Algebraic Analysis, PP. 129-139, Acad. Prss, 1988
- [12] S.L. Woronowicz, Commun. Math. Phys. 122 (1989) 125-170.
- [13] V.N. Tolstoy and S.M. Khoroshkin, Func. Anal. Appl. 26 (1992) 69-71;
S.M. Khoroshkin, A.A Stolin and V.N. Tolstoy, Mod. Phys. Lett A10 (1995) 1375-1392.
- [14] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, Commun. Math. Phys. 177 (1996) 381-398; ibid. 190 (1997) 247-278; and ibid 200 (1999) 297-324.
- [15] G. W. Delius, M. D. Gould and Y. Z. Zhang, Nucl. Phys. B **432** (1994) 377 [arXiv:hep-th/9405030].
- [16] V.V. Bazhanov and Yu.G. Stroganov, J. Stat. Phys. **59** (1990) 799.
- [17] L.N. Lipatov, *High-energy asymptotics of multicolor QCD and exactly solvable lattice models* Padova preprint DFPD-93-TH-70B; and JETP Lett. B342 (1994)596.
- [18] L. D. Faddeev and G. P. Korchemsky, Phys. Lett. B **342** (1995) 311 [arXiv:hep-th/9404173].
- [19] L. N. Lipatov, Phys. Lett. B **309** (1993) 394.
- [20] G. P. Korchemsky, Nucl. Phys. B **443** (1995) 255 [arXiv:hep-ph/9501232].
- [21] G. P. Korchemsky, Nucl. Phys. B **462** (1996) 333 [arXiv:hep-th/9508025].
- [22] L. N. Lipatov, Nucl. Phys. B **548** (1999) 328 [arXiv:hep-ph/9812336].
- [23] H. J. De Vega and L. N. Lipatov, Phys. Rev. D**64** (2001) 114019, arXiv:hep-ph/0107225.

- [24] S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, Nucl. Phys. B **617** (2001) 375, arXiv:hep-th/0107193.
- [25] D. R. Karakhanian and R. Kirschner, Fortsch. Phys. **48** (2000) 139 [arXiv:hep-th/9902031].
- [26] S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, Nucl. Phys. B **566** (2000) 203 [arXiv:hep-ph/9909539].
- [27] V. M. Braun, S. E. Derkachov and A. N. Manashov, Phys. Rev. Lett. **81** (1998) 2020 [arXiv:hep-ph/9805225].
- [28] S. E. Derkachov and R. Kirschner, Phys. Rev. D **64** (2001) 074013 [arXiv:hep-ph/0101174].
- [29] S. Derkachov, D. Karakhanian and R. Kirschner, Nucl. Phys. B **583** (2000) 691 [arXiv:nlin.si/0003029].
- [30] S. E. Derkachov, D. Karakhanian and R. Kirschner, Nucl. Phys. B **618** (2001) 589, arXiv:nlin.si/0102024.
- [31] P.P. Kulish and E.K. Sklyanin, Zap. Nauchn. Semin. LOMI 95 (1980) 129.
- [32] V.O. Tarasov, L.A. Takhtadjan and L.D. Faddeev, Theor. Math. Phys. 57 (1983) 163-181.
- [33] E.K. Sklyanin, "Quantum Inverse Scattering Method", in *Quantum Groups and Quantum Integrable Systems*, (Nankai lectures), ed. Mo-Lin Ge, pp. 63-97, World Scientific Publ., Singapore 1992, [hep-th/9211111]
- [34] L.D. Faddeev, Les Houches lectures 1995, hep-th/9605187.
- [35] G. Gasper and M. Rahman, *Basic hypergeometric series*, Cambridge University Press, 1990.